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By

JOHN CARSTOIU

Science for Industry, Inc., Brookline, Massachusetts

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MAGNETOHYDRODYNAMIC WAVES IN A CONSTANT DIPOLE MAGNETIC FIELD*

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SCIENCE FOR INDUSTRY, INC., BROOKLINE, MASSACHUSETTS

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1. *Introduction.*—Dungey¹ in his remarkable report of 1954 has discussed in some detail the electrodynamic behavior of the outer atmosphere in the presence of a constant dipole magnetic field. We shall here approach the problem from a different point of view, however, concentrating our attention (as we did in our previous studies^{2, 3}) on the vorticity field and the current density.

Consider an infinite mass of an electrically conducting fluid at rest, embedded in a constant dipole magnetic field \mathbf{H} . To simplify the discussion, take the conductivity as infinite (an approximation justified for a large-scale disturbance) and assume the fluid to be a homogeneous incompressible material. Assume that as a result of a perturbation, a velocity field \mathbf{v} is produced in a certain region and that the magnetic field becomes $\mathbf{H} + \mathbf{h}$. The amplitude is assumed to be small enough for nonlinear terms to be neglected. We propose to investigate the magnetohydrodynamic behavior of the fluid in terms of generalized Alfvén waves by means of vorticity and current density. The equations obtained are complicated, however, and solutions will be discussed only for large distances from the center of the dipole. We shall conclude this note with an appendix where the geometry of lines of force is briefly discussed; the results obtained there are not all original, but are essential to an understanding of the subject.

2. *Fundamental Equations.*—The equations relevant to the problem are

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\text{grad } p + \mu_e \mathbf{j} \times \mathbf{H}, \quad (1)$$

$$\frac{\partial \mathbf{h}}{\partial t} = (\mathbf{H} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{H}, \quad (2)$$

$$\text{div } \mathbf{v} = 0, \quad (3)$$

$$\text{div } \mathbf{h} = 0, \quad (4)$$

where ρ_0 is the density (a constant), p the pressure, μ_e the permeability, $\mathbf{j} = (1/4\pi) \text{curl } \mathbf{h}$, and the electromagnetic variables are measured in emu; the condition $\partial \mathbf{H} / \partial t = 0$ (constant dipole) has been used in equation (2).

Since $\text{curl } \mathbf{H} = 0$, equation (1) can be rewritten as follows:

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\text{grad} \left(p + \frac{\mu_e \mathbf{H} \cdot \mathbf{h}}{4\pi} \right) + \frac{\mu_e}{4\pi} [(\mathbf{H} \cdot \nabla) \mathbf{h} + (\mathbf{h} \cdot \nabla) \mathbf{H}]. \quad (5)$$

Taking the curl of terms of equations (5) and (2) we obtain

$$2\rho_0 \frac{\partial \boldsymbol{\omega}}{\partial t} = \frac{\mu_e}{4\pi} \text{curl} [(\mathbf{H} \cdot \nabla) \mathbf{h} + (\mathbf{h} \cdot \nabla) \mathbf{H}], \quad (6)$$

$$4\pi \frac{\partial \mathbf{j}}{\partial t} = \text{curl} [(\mathbf{H} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{H}], \quad (7)$$

where $\boldsymbol{\omega} = (1/2) \text{curl } \mathbf{v}$ is the vorticity.

We have

$$\begin{aligned} \text{curl} [(\mathbf{H} \cdot \nabla) \mathbf{h} + (\mathbf{h} \cdot \nabla) \mathbf{H}] &= 4\pi (\mathbf{H} \cdot \nabla) \mathbf{j} \\ &+ \text{grad } h_x \times \frac{\partial \mathbf{H}}{\partial x} + \text{grad } h_y \times \frac{\partial \mathbf{H}}{\partial y} + \text{grad } h_z \times \frac{\partial \mathbf{H}}{\partial z} \\ &+ \text{grad } H_x \times \frac{\partial \mathbf{h}}{\partial x} + \text{grad } H_y \times \frac{\partial \mathbf{h}}{\partial y} + \text{grad } H_z \times \frac{\partial \mathbf{h}}{\partial z}. \end{aligned} \quad (8)$$

After some calculation, we obtain

$$\text{curl} [(\mathbf{H} \cdot \nabla) \mathbf{h} + (\mathbf{h} \cdot \nabla) \mathbf{H}] = 4\pi [(\mathbf{H} \cdot \nabla) \mathbf{j} - (\mathbf{j} \cdot \nabla) \mathbf{H}], \quad (9)$$

where the condition $\text{curl } \mathbf{H} = 0$ has been used.

Hence, equation (6) can be written

$$2\rho_0 \frac{\partial \boldsymbol{\omega}}{\partial t} = \mu_e [(\mathbf{H} \cdot \nabla) \mathbf{j} - (\mathbf{j} \cdot \nabla) \mathbf{H}]. \quad (10)$$

On the other hand,

$$\begin{aligned} \text{curl} [(\mathbf{H} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{H}] &= 2(\mathbf{H} \cdot \nabla) \boldsymbol{\omega} \\ &+ \text{grad } H_x \times \frac{\partial \mathbf{v}}{\partial x} + \text{grad } H_y \times \frac{\partial \mathbf{v}}{\partial y} + \text{grad } H_z \times \frac{\partial \mathbf{v}}{\partial z} \end{aligned}$$

$$- \left(\text{grad } v_x \times \frac{\partial \mathbf{H}}{\partial x} + \text{grad } v_y \times \frac{\partial \mathbf{H}}{\partial y} + \text{grad } v_z \times \frac{\partial \mathbf{H}}{\partial z} \right). \quad (11)$$

However, in this case there is no simple way to write vectorially equation (7) in a compact form; we have for components the following equations

$$\left. \begin{aligned} 2\pi \frac{\partial j_x}{\partial t} &= [(\mathbf{H} \cdot \nabla) \omega]_x \\ &+ \frac{\partial H_x}{\partial y} e_{31} + \frac{\partial H_y}{\partial y} e_{32} + \frac{\partial H_z}{\partial y} e_{33} - \left(\frac{\partial H_x}{\partial z} e_{21} + \frac{\partial H_y}{\partial z} e_{22} + \frac{\partial H_z}{\partial z} e_{23} \right), \\ 2\pi \frac{\partial j_y}{\partial t} &= [(\mathbf{H} \cdot \nabla) \omega]_y \\ &+ \frac{\partial H_x}{\partial z} e_{11} + \frac{\partial H_y}{\partial z} e_{12} + \frac{\partial H_z}{\partial z} e_{13} - \left(\frac{\partial H_x}{\partial x} e_{31} + \frac{\partial H_y}{\partial x} e_{32} + \frac{\partial H_z}{\partial x} e_{33} \right), \\ 2\pi \frac{\partial j_z}{\partial t} &= [(\mathbf{H} \cdot \nabla) \omega]_z \\ &+ \frac{\partial H_x}{\partial x} e_{12} + \frac{\partial H_y}{\partial x} e_{22} + \frac{\partial H_z}{\partial x} e_{23} - \left(\frac{\partial H_x}{\partial y} e_{11} + \frac{\partial H_y}{\partial y} e_{12} + \frac{\partial H_z}{\partial y} e_{13} \right), \end{aligned} \right\} \quad (12)$$

where e_{ij} is the rate of deformation: $e_{ij} = 1/2[(\partial v_i/\partial x_j + \partial v_j/\partial x_i)]$, and where again the condition $\text{curl } \mathbf{H} = 0$ has been used.

3. *Propagation at Large Distances.*—These equations are rather complicated. We may simplify them by observing that the derivatives of the components of the dipole magnetic field are of the order of r^{-4} , while these components themselves are of the order of r^{-3} . Therefore, for r sufficiently large, we may neglect the term $(\mathbf{j} \cdot \nabla) \mathbf{H}$ in equation (10), and similarly we may neglect all terms such as $(\partial H_x/\partial y)e_{31}$, etc. in equations (12). Hence for r sufficiently large, equations (10) and (12) reduce to

$$2\rho_0 \frac{\partial \omega}{\partial t} = \mu_e \left(H_x \frac{\partial \mathbf{j}}{\partial x} + H_y \frac{\partial \mathbf{j}}{\partial y} + H_z \frac{\partial \mathbf{j}}{\partial z} \right), \quad (13)$$

and

$$2\pi \frac{\partial \mathbf{j}}{\partial t} = H_x \frac{\partial \omega}{\partial x} + H_y \frac{\partial \omega}{\partial y} + H_z \frac{\partial \omega}{\partial z}. \quad (14)$$

In the second place, we have along a line of force

$$H_x = H \frac{dx}{ds}, \quad H_y = H \frac{dy}{ds}, \quad H_z = H \frac{dz}{ds}, \quad (15)$$

where H is the magnitude of the dipole magnetic field and s is the element of length of a line of force.

Hence, along a line of force, equations (13) and (14) can be written

$$2\rho_0 \frac{\partial \omega}{\partial t} = \mu_e H \left(\frac{\partial \mathbf{j}}{\partial x} \frac{dx}{ds} + \frac{\partial \mathbf{j}}{\partial y} \frac{dy}{ds} + \frac{\partial \mathbf{j}}{\partial z} \frac{dz}{ds} \right), \quad (16)$$

and

$$2\pi \frac{\partial \mathbf{j}}{\partial t} = H \left(\frac{\partial \omega}{\partial x} \frac{dx}{ds} + \frac{\partial \omega}{\partial y} \frac{dy}{ds} + \frac{\partial \omega}{\partial z} \frac{dz}{ds} \right), \quad (17)$$

that is

$$2\rho_0 \frac{\partial \omega}{\partial t} = \mu_e H \frac{\partial \mathbf{j}}{\partial s}, \quad (18)$$

$$2\pi \frac{\partial \mathbf{j}}{\partial t} = H \frac{\partial \omega}{\partial s}. \quad (19)$$

Observing that $\partial H / \partial t = 0$ (constant dipole), we obtain by cross-differentiation

$$\frac{\partial^2 \omega}{\partial t^2} = A^2 \frac{\partial^2 \omega}{\partial s^2} + \frac{1}{2} \frac{dA^2}{ds} \frac{\partial \omega}{\partial s}, \quad (20)$$

$$\frac{\partial^2 \mathbf{j}}{\partial t^2} = A^2 \frac{\partial^2 \mathbf{j}}{\partial s^2} + \frac{1}{2} \frac{dA^2}{ds} \frac{\partial \mathbf{j}}{\partial s}, \quad (21)$$

where

$$\begin{aligned} A^2 &= \frac{\mu_e H^2}{4\pi\rho_0} = \frac{\mu_e M^2}{4\pi\rho_0} \frac{\cos^2 \lambda}{r^6} (1 + 4 \tan^2 \lambda) \\ &= \gamma^2 \frac{1 + 4 \tan^2 \lambda}{r_0^6 \cos^{10} \lambda}, \end{aligned} \quad (22)$$

and where we put $\gamma^2 = \mu_e M^2 / 4\pi r_0$; the other notations are given in appendix. We achieve the reduction of these equations by taking instead of s the magnetic latitude, λ , as independent variable. We have

$$\frac{\partial \omega}{\partial s} = \frac{\partial \omega}{\partial \lambda} \frac{d\lambda}{ds}, \quad (23)$$

$$\frac{\partial^2 \omega}{\partial s^2} = \frac{\partial^2 \omega}{\partial \lambda^2} \left(\frac{d\lambda}{ds} \right)^2 + \frac{\partial \omega}{\partial \lambda} \frac{d^2 \lambda}{ds^2}, \quad (24)$$

and

$$\frac{d^2 \lambda}{ds^2} = \frac{d\lambda}{ds} \frac{d}{d\lambda} \left(\frac{d\lambda}{ds} \right). \quad (25)$$

Now (see appendix)

$$\frac{d\lambda}{ds} = \frac{1}{r_0 \cos^2 \lambda \sqrt{1 + 4 \tan^2 \lambda}}; \quad (26)$$

therefore, we have

$$\frac{d^2 \lambda}{ds^2} = \frac{2(2 \tan^2 \lambda - 1) \tan \lambda}{r_0^2 \cos^4 \lambda (1 + 4 \tan^2 \lambda)^2}. \quad (27)$$

On the other hand,

$$\begin{aligned} \frac{dA^2}{ds} \frac{\partial \omega}{\partial s} &= \frac{dA^2}{d\lambda} \left(\frac{d\lambda}{ds} \right)^2 \frac{\partial \omega}{\partial \lambda} \\ &= \frac{6\gamma^2(3 + 8 \tan^2 \lambda) \tan \lambda}{r_0^8 \cos^{14} \lambda (1 + 4 \tan^2 \lambda)} \frac{\partial \omega}{\partial \lambda}. \end{aligned} \quad (28)$$

Substitution of these values in equation (20) gives

$$\frac{\partial^2 \omega}{\partial t^2} = \frac{\gamma^2}{r_0^8 \cos^{14} \lambda} \left(\frac{\partial^2 \omega}{\partial \lambda^2} + 7 \tan \lambda \frac{\partial \omega}{\partial \lambda} \right). \quad (29)$$

We have, of course, an identical equation for j .

Supposing that

$$\omega(\lambda, t) = C e^{i\alpha t} \omega_1(\lambda), \quad (30)$$

we get

$$\frac{d^2 \omega_1}{d\lambda^2} + 7 \tan \lambda \frac{d\omega_1}{d\lambda} + \beta^2 r_0^8 \cos^{14} \lambda \omega_1 = 0, \quad (31)$$

where $\beta^2 = \alpha^2 / \gamma^2$.

4. *Integral Equations for Vorticity and Current Density.*—Equation (31) can be transformed into an integral equation similar to that given by Dungey (*loc. cit.*, p. 33). In order to do this, we use the identity

$$\begin{aligned} \frac{d}{d\lambda} \left[\sec \lambda \frac{d}{d\lambda} (\omega_1 \sec^3 \lambda) \right] \\ = \sec^4 \lambda \left[\frac{d^2 \omega_1}{d\lambda^2} + 7 \tan \lambda \frac{d\omega_1}{d\lambda} + 3(1 + 5 \tan^2 \lambda) \omega_1 \right]. \end{aligned} \quad (32)$$

By virtue of (32), equation (31) can be written

$$\frac{d}{d\lambda} \left[\sec \lambda \frac{d}{d\lambda} (\omega_1 \sec^3 \lambda) \right] = [3(1 + 5 \tan^2 \lambda) \sec \lambda - \beta^2 r_0^8 \cos^{13} \lambda] \omega_1 \sec^3 \lambda, \quad (33)$$

or, by putting $\Omega = \omega_1 \sec^3 \lambda$,

$$\frac{d}{d\lambda} \left(\sec \lambda \frac{d\Omega}{d\lambda} \right) = [3(1 + 5 \tan^2 \lambda) \sec \lambda - \beta^2 r_0^8 \cos^{13} \lambda] \Omega. \quad (34)$$

Assuming that for $\lambda = 0$, $d\Omega/d\lambda = 0$, i.e., $d\omega_1/d\lambda = 0$, we obtain the following integral equation for $\Omega(\lambda)$:

$$\begin{aligned} \Omega(\lambda) &= \Omega(0) \\ &+ \int_0^\lambda \cos \lambda' d\lambda' \int_0^{\lambda'} [3(1 + 5 \tan^2 \lambda'') \sec \lambda'' - \beta^2 r_0^8 \cos^{13} \lambda''] \Omega(\lambda'') d\lambda'', \end{aligned} \quad (35)$$

which, curiously enough, has the same form as the equation given by Dungey (*loc. cit.* p. 33) but is of vectorial character and includes an additional term $3(1 + 5 \tan^2 \lambda) \sec \lambda$; the variables and assumptions used to arrive at this result differ radically from those used by Dungey.

The quantity $J(\lambda) = j_1(\lambda) \sec^3 \lambda$ verifies the same equation (35). Equation (35) may be integrated by successive approximations.

Appendix.—The geometry of lines of force: As is well known, a dipole magnetic field has components

$$H_x = -M \frac{3xz}{r^5}, H_y = -M \frac{3yz}{r^5}, H_z = -M \frac{r^2 - 3z^2}{r^5}, \quad (36)$$

where M is the magnetic moment of the dipole, $r^2 = x^2 + y^2 + z^2$, and the sign is chosen such that H_z is positive in the xy -plane, which is the equatorial plane of the dipole; this requires that the dipole have its negative pole directed upward (see Burgers⁴). The differential equations of lines of force are

$$\frac{dx}{-3xz} = \frac{dy}{-3yz} = \frac{dz}{x^2 + y^2 - 2z^2}. \quad (37)$$

Integration of these equations gives

$$\left. \begin{aligned} x^{2/3} &= Br, \\ y &= Cx, \end{aligned} \right\} \quad (38)$$

where B and C are constants of integration.

We now introduce the polar coordinates

$$\left. \begin{aligned} x &= r \cos \varphi \cos \lambda, \\ y &= r \sin \varphi \cos \lambda, \\ z &= r \sin \lambda, \end{aligned} \right\} \quad (39)^*$$

where λ designates the magnetic latitude. In these coordinates, the first equation (38) becomes

$$\cos^2 \varphi \cos^2 \lambda = B^2 r \quad (40)$$

To eliminate φ , we write

$$\frac{y}{x} = C = \tan \varphi. \quad (44)$$

Hence equation (40) becomes

$$r = r_0 \cos^2 \lambda, \quad (42)$$

which is a well-known result; it is obvious that the constant r_0 is the value of r for $\lambda = 0$ (in equatorial plane).

In Cartesian coordinates, we have the following parametric equations of lines of force

$$\left. \begin{aligned} x &= \frac{r \cos \lambda}{\sqrt{C^2 + 1}}, \\ y &= \frac{Cr \cos \lambda}{\sqrt{C^2 + 1}}, \\ z &= r \sin \lambda. \end{aligned} \right\} \quad (43)$$

To calculate the linear element of these lines, we may use either equation (42) and then

$$ds^2 = dr^2 + r^2 d\lambda^2, \quad (44)$$

or the parametric equations (43) and then we have

$$ds^2 = dx^2 + dy^2 + dz^2. \quad (45)$$

The result is

$$ds = r d\lambda \sqrt{1 + 4 \tan^2 \lambda}. \quad (46)$$

The tangent to a line of force is defined by

$$\left. \begin{aligned} \alpha &= \frac{dx}{ds} = -\frac{3}{\sqrt{C^2 + 1}} \frac{\sin \lambda}{\sqrt{1 + 4 \tan^2 \lambda}}, \\ \beta &= \frac{dy}{ds} = -\frac{3C}{\sqrt{C^2 + 1}} \frac{\sin \lambda}{\sqrt{1 + 4 \tan^2 \lambda}}, \\ \gamma &= \frac{dz}{ds} = \frac{(1 - 2 \tan^2 \lambda) \cos \lambda}{\sqrt{1 + 4 \tan^2 \lambda}}. \end{aligned} \right\} \quad (47)$$

The radius of curvature R is given by

$$\frac{1}{R^2} = \left(\frac{d\alpha}{ds} \right)^2 + \left(\frac{d\beta}{ds} \right)^2 + \left(\frac{d\gamma}{ds} \right)^2. \quad (48)$$

After some calculations we find

$$\frac{1}{R^2} = \frac{9(1 + 2 \tan^2 \lambda)^2}{r^2(1 + 4 \tan^2 \lambda)^3}. \quad (49)$$

In the equatorial plane,

$$R = \frac{r}{3} = \frac{r_0}{3}. \quad (50)$$

As a verification of result (49), we may use the formula

$$R = \frac{(r^2 + r'^2)^{3/2}}{r^2 + 2r'^2 - rr''}, \quad (51)$$

where r is given by (42).

The principal normal, which in our case reduces to the normal of lines of force since these are plane curves, is determined by

$$\left. \begin{aligned} \alpha_1 &= R \frac{d\alpha}{ds} = -\frac{\cos \lambda}{\sqrt{C^2 + 1}} \frac{1 - 2 \tan^2 \lambda}{\sqrt{1 + 4 \tan^2 \lambda}}, \\ \beta_1 &= R \frac{d\beta}{ds} = -\frac{C \cos \lambda}{\sqrt{C^2 + 1}} \frac{1 - 2 \tan^2 \lambda}{\sqrt{1 + 4 \tan^2 \lambda}}, \\ \gamma_1 &= R \frac{d\gamma}{ds} = -\frac{3 \sin \lambda}{\sqrt{1 + 4 \tan^2 \lambda}}. \end{aligned} \right\} \quad (52)$$

The binormal of the lines of force, of course, is the normal to the planes $y = Cx$; therefore, its direction cosines are

$$\alpha_2 = \frac{C}{\sqrt{C^2 + 1}}, \quad \beta_2 = -\frac{1}{\sqrt{C^2 + 1}}, \quad \gamma_2 = 0. \quad (53)$$

These values may also be calculated from the formulas

$$\alpha_2 = \beta \gamma_1 - \gamma \beta_1, \text{ etc.}$$

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¹ Dungey, J. W., in *Electrodynamics of the Outer Atmosphere*, Pennsylvania State University, Ionosphere Research Laboratory, Scientific Report No. 69 (1954).

² Carstoiu, J., these PROCEEDINGS, 46, 131-136 (1960).

³ *Ibid.*, 47, 891-898 (1961).

⁴ Burgers, J. W., in *Lectures on Fluid Mechanics* by Sydney Goldstein (New York: Interscience Publishers, 1960), p. 273.